OPERATIONAL RESOLUTIONS AND STATE TRANSITIONS IN A CATEGORICAL SETTING[1]

Bob Coecke and Isar Stubbe

FUND-DWIS, Free University of Brussels,
Pleinlaan 2, B-1050 Brussels, Belgium.
bocoecke@vub.ac.be

and

AGEL-MAPA, Université Catholique de Louvain,
Ch. du Cyclotron 2, B-1348 Louvain-La-Neuve, Belgium.
i.stubbe@agel.ucl.ac.be

We define a category with as objects operational resolutions and with as morphisms — not necessarily deterministic — state transitions. We study connections with closure spaces and join-complete lattices and sketch physical applications related to evolution and compoundness. An appendix with preliminaries on quantaloids is included.

Key words: state and property transitions, closure space, complete lattice, quantales and quantaloids.

1. INTRODUCTION

The core of the mathematical development in this paper consists of lifting the — categorically — equivalent descriptions of physical systems by a (i) 'state space' or a (ii) 'property lattice' — see [14,20,25,26] — to an asymmetrical — i.e., not anymore isomorphic — duality on the level of:

(i)*bis* *'possible state' transitions — 'possible' in the sense that an arbitrary initial state is mapped onto all possible outcome states for this*

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particular 'not necessarily deterministic' transition, i.e., we consider the propagation of those states that are "possibly true" with respect to the indeterministic nature of this transition,

and,

(ii)[bis] *'definite property' transitions — 'definite' in the sense that we consider the propagation of only those properties that are "true with certainty", even when the transition is not deterministic.*

However, our mathematical setup is somewhat more general than the one in [14,20,25,26] since we consider any set equipped with a closure operator as a state space and any complete lattice as a property lattice. We reach this goal by demanding that for a 'possible state transition', a 'definite property transition' is well-defined and preserves the lattice join. It can indeed be physically be motivated that property transitions should be described by join preserving maps, or equivalently in the case of complete orthomodular property lattices, by a complete Baer*-semigroup of hemimorphisms [3,11,12,13,15,29]. We now briefly sketch a physical argumentation for this fact along the lines of [15][1]. With an evolution from time t_0 to time t_1 we can associate a map $f^* : \mathcal{L} \to \mathcal{L}$ with $a_0 = f^*(a_1)$ being the cause of a_1 in the property lattice $\mathcal{L}$ on t_0, that is, a_0 is the weakest property in $\mathcal{L}$ on time t_0 whose actuality — read 'being true' — guarantees the actuality of a_1 on time t_1. The fact that the lattice meet is nothing else than the semantic conjunction [27] then implies that f^* preserves non-empty meets. As a consequence, it has a join preserving Galois dual [4,17,21]:

$$f : [0, f^*(1)] \to \mathcal{L} : a_0 \mapsto \wedge \{a_1 \in \mathcal{L} \mid a_0 \leq f^*(a_1)\} \qquad (1)$$

that exactly expresses the propagation of the properties: f maps an arbitrary property a_0 to the strongest one — expressed as a meet — whose actuality is implied by the actuality of a_0. Conversely, to any such join preserving map f, expressing propagation, we can associate a meet preserving map f^* that expresses the physically justifiable assignation of temporal causes for that evolution. In [15] however, only so called "strong deterministic evolutions" that send atoms to atoms have been considered, as such excluding the description of indeterministic state transitions. In our approach we formally integrate indeterministic transitions by considering maps on powersets of a state space, as it will be discussed in more detail in the 5th section of this paper. Besides this indeterministic aspect we will also consider state transitions with non-equal domain and codomain for essentially two reasons: (i) some externally induced state transitions might correspond with an actual change of the state space of the system; (ii) maps between different property lattices provide an appropriate structure for the description of mutually induced state transitions between individual entities within a compound system — a proof for the existence of such a representation for compound quantum systems can be found in [7,9]. For more details on this aspect we refer to the 6th section of this paper. The

main motivation for a categorical treatment of state transitions is the observation that they compose in a natural way by consecutive application, and that composition of morphisms is exactly the structural ingredient that constitutes a category [1,5,19]. As such, it is in our particular setup very natural to express state transitions as morphisms of a category, where the objects describe the states and properties of a physical system. Functorality of maps between categories then expresses preservation of consecutive application, in our case by coupling 'possible state transitions' and corresponding 'definite property transitions'.

2. OPERATIONAL RESOLUTIONS

We begin by defining the objects of our categories.

Definition 1. *For a given set Σ, an 'operational resolution' is defined as a map $\mathcal{C}_{pr} : \mathcal{P}(\Sigma) \to \mathcal{L}$ with as domain the powerset of Σ and as codomain a poclass[2] $(\mathcal{L}, \leq)$, such that for all $T, T', T_i \in \mathcal{P}(\Sigma)$:*

$$
\begin{align}
T \subseteq T' & \quad\Rightarrow\quad \mathcal{C}_{pr}(T) \leq \mathcal{C}_{pr}(T') \tag{2} \\
\forall i : \mathcal{C}_{pr}(T_i) \leq \mathcal{C}_{pr}(T) & \quad\Rightarrow\quad \mathcal{C}_{pr}(\cup_i T_i) \leq \mathcal{C}_{pr}(T) \tag{3} \\
T \neq \emptyset & \quad\Rightarrow\quad \mathcal{C}_{pr}(T) \neq \mathcal{C}_{pr}(\emptyset) \tag{4}
\end{align}
$$

The chosen axioms can be verified physically by interpreting $\mathcal{C}_{pr}$ as assigning to a collection of 'possible states' $T \in \mathcal{P}(\Sigma)$ the smallest — i.e., strongest — 'definite property' $\mathcal{C}_{pr}(T) \in \mathcal{L}$ physically implied by every state $p \in T$, i.e, referring to the terminology of [14,20,25,26], it is the conjunction of all properties that are always actual whenever at least one $p \in T$ is actual — the existence of such a conjunction follows from Theorem 1. Let us first recall some basic results on such operational resolutions [11]. The image of $\mathcal{C}_{pr}$, which is a subset of the class $\mathcal{L}$ and thus inherits the partial order $\leq$, is a complete lattice with, for any $\{T_i\}_i \subseteq \mathcal{P}(\Sigma)$, $\vee_i \mathcal{C}_{pr}(T_i) = \mathcal{C}_{pr}(\cup_i T_i)$, bottom element $\mathcal{C}_{pr}(\emptyset)$ and top element $\mathcal{C}_{pr}(\Sigma)$. Also, $\{\mathcal{C}_{pr}(p) \mid p \in \Sigma\}$ is an order generating set of $im(\mathcal{C}_{pr})$, in the sense that $\forall T \in \mathcal{P}(\Sigma) : \mathcal{C}_{pr}(T) = \vee_{t \in T} \mathcal{C}_{pr}(t)$. Given a set X, an operator $\mathcal{C} : \mathcal{P}(X) \to \mathcal{P}(X)$ on the powerset of X is called 'closure operator on X' if the following are met: (C1): $T \subseteq \mathcal{C}(T)$; (C2): $T \subseteq T' \Rightarrow \mathcal{C}(T) \subseteq \mathcal{C}(T')$; (C3): $\mathcal{C}(\mathcal{C}(T)) = \mathcal{C}(T)$. A closure is called T_0 if moreover $\mathcal{C}(\emptyset) = \emptyset$ and $\mathcal{C}(\{x\}) = \mathcal{C}(\{y\}) \Rightarrow x = y$ for any $x, y \in X$. It is called T_1 if $\mathcal{C}(\emptyset) = \emptyset$ and $\mathcal{C}(\{x\}) = \{x\}$ for any $x \in X$. We have that every operational resolution $\mathcal{C}_{pr} : \mathcal{P}(\Sigma) \to \mathcal{L}$ factors into a closure operator $\mathcal{C} : \mathcal{P}(\Sigma) \to \mathcal{F}(\Sigma) \subseteq \mathcal{P}(\Sigma)$ with $\mathcal{C}(\emptyset) = \emptyset$, and an embedding of the poset of $\mathcal{C}$-closed subsets $\mathcal{F}(\Sigma)$ into the poclass $\mathcal{L}$, $\theta : \mathcal{F}(\Sigma) \to \mathcal{L}$, such that $\mathcal{F}(\Sigma) \cong im(\theta) = im(\mathcal{C}_{pr})$ as lattices. The closure factor is given by $\mathcal{C}(T) = \cup\{S \in \mathcal{P}(\Sigma) \mid \mathcal{C}_{pr}(S) = \mathcal{C}_{pr}(T)\}$,

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which can be rewritten as $\mathcal{C}(T) = \{t \in \Sigma \mid \mathcal{C}_{pr}(t) \leq \mathcal{C}_{pr}(T)\}$. The prescription of the embedding θ is $\theta(F) = \mathcal{C}_{pr}(F)$, for any $F \in \mathcal{F}(\Sigma)$. It can be verified that this factorization is unique. Conversely to the factorization, any closure space $(\Sigma, \mathcal{C})$ for which $\mathcal{C}(\emptyset) = \emptyset$ and any embedding of the poset of $\mathcal{C}$-closed subsets of Σ in a poclass $\mathcal{L}$, say $\theta : \mathcal{F}(\Sigma) \to \mathcal{L}$, such that $\mathcal{F}(\Sigma) \cong im(\theta)$ as lattices, uniquely define an operational resolution, namely $\mathcal{C}_{pr} = \theta \circ \mathcal{C}$. Referring to these results, we state the following theorem.

Theorem 1. *Given a set Σ and a poclass $\mathcal{L}$, a map $\mathcal{C}_{pr} : \mathcal{P}(\Sigma) \to \mathcal{L}$ is an operational resolution if and only if it factors uniquely into a closure $\mathcal{C}$ with $\mathcal{C}(\emptyset) = \emptyset$, and an embedding θ of the poset of $\mathcal{C}$-closed subsets $\mathcal{F}(\Sigma)$ into the poclass $\mathcal{L}$ such that the image of θ, inheriting the order from $\mathcal{L}$, is a complete lattice that is isomorphic to $\mathcal{F}(\Sigma)$.*

This means that $\mathcal{C}_{pr} : \mathcal{P}(\Sigma) \to \mathcal{L}$ can be characterized either by the triple $(\Sigma, \mathcal{L}, \mathcal{C}_{pr})$ or by the quadruple $(\Sigma, \mathcal{L}, \mathcal{C}, \theta)$.

Next we want to elaborate on "how an operational resolution orders and separates points", much in analogy to closure operators. These considerations will lead to the notion of a 'canonical resolution'.

Definition 2. *An operational resolution $\mathcal{C}_{pr} : \mathcal{P}(\Sigma) \to \mathcal{L}$ is a T_0-resolution if $\forall p, q \in \Sigma : \mathcal{C}_{pr}(p) = \mathcal{C}_{pr}(q) \Rightarrow p = q$; it is a T_1-resolution if $\forall p, q \in \Sigma : \mathcal{C}_{pr}(p) \leq \mathcal{C}_{pr}(q) \Rightarrow p = q$.*

It can be verified straightforwardly, and it justifies the terminology in Definition 2, that $\mathcal{C}_{pr}$ is T_0 (T_1) if and only if its closure factor $\mathcal{C}$ is T_0 (T_1). Concerning an operational resolution $\mathcal{C}_{pr} : \mathcal{P}(\Sigma) \to \mathcal{L}$, we further introduce the following notations, for $p, q \in \Sigma$:

$$\left\{ \begin{array}{lcl} p \lhd_{pr} q & \Leftrightarrow & \mathcal{C}_{pr}(p) < \mathcal{C}_{pr}(q) \\ p =_{pr} q & \Leftrightarrow & \mathcal{C}_{pr}(p) = \mathcal{C}_{pr}(q) \\ p \unlhd_{pr} q & \Leftrightarrow & p \lhd_{pr} q \ \ or \ \ p =_{pr} q \end{array} \right.$$

This defines a preordered set $(\Sigma, \unlhd_{pr})$. It obviously follows that $\mathcal{C}_{pr}$ is T_0 if and only if $[p =_{pr} q \Rightarrow p = q]$ for all $p, q \in \Sigma$ and that $\mathcal{C}_{pr}$ is T_1 if and only if $[p \unlhd_{pr} q \Rightarrow p = q]$ for all $p, q \in \Sigma$. The following examples prove their importance further in this text.

Example 1. *If Σ is a 'full set of states' [2, 26] for a complete lattice $\mathcal{L}$ — i.e., Σ is a subset of $\mathcal{L}$, not containing the bottom element, such that $t = \vee \{a \in \Sigma \mid a \leq t\}$ for all $t \in \mathcal{L}$ — then $\mathcal{C}_{pr} : \mathcal{P}(\Sigma) \to \mathcal{L} : T \mapsto \vee T$ is an operational resolution. Σ inherits order from $\mathcal{L}$, and this order coincides with $\unlhd_{pr}$ (from a slightly different viewpoint we could also say that this operational resolution "recuperates" the a priori order on Σ, which is inherited from $\mathcal{L}$ through set-inclusion). This operational resolution is always T_0. If $\mathcal{L}$ is atomistic and Σ is its set of atoms, then and only then it is T_1.*

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This example exhibits how the notion of "operational resolution" generalizes the state/property duality as it is put forward in [20].

Example 2. *Given any closure $\mathcal{C}$ on a set Σ such that $\mathcal{C}(\emptyset) = \emptyset$, $\mathcal{C} : \mathcal{P}(\Sigma) \to \mathcal{F}(\Sigma)$ — where $\mathcal{F}(\Sigma)$ is the family of $\mathcal{C}$-closed subsets of Σ — defines an operational resolution. This operational resolution is obviously T_0 (T_1) whenever $\mathcal{C}$ is so as closure.*

By definition of $\unlhd_{pr}$, the surjection $\Sigma \to \{\mathcal{C}_{pr}(p) \mid p \in \Sigma\} : p \mapsto \mathcal{C}_{pr}(p)$ preserves $\unlhd_{pr}$. This map is injective, thus bijective exactly for T_0 resolutions. We have that $\{\mathcal{C}_{pr}(p) \mid p \in \Sigma\} \subseteq im(\mathcal{C}_{pr}) \setminus \{\mathcal{C}_{pr}(\emptyset)\}$, but if this inclusion 'saturates' to an equality, then we have a very particular kind of operational resolution at hand.

Definition 3. *$\mathcal{C}_{pr} : \mathcal{P}(\Sigma) \to \mathcal{L}$ will be called 'saturated' if the map $\Sigma \to im(\mathcal{C}_{pr}) \setminus \{\mathcal{C}_{pr}(\emptyset)\} : p \mapsto \mathcal{C}_{pr}(p)$ is surjective. An operational resolution that is saturated and T_0, will be called 'canonical'.*

If $\mathcal{C}_{pr}$ is canonical then, and only then, we have isomorphic lattices $(\Sigma \cup \{0\}, \unlhd_{pr}) \cong (im(\mathcal{C}_{pr}), \leq)$ where we define that $0 \unlhd_{pr} p$ for all $p \in \Sigma$, and where $\leq$ on $im(\mathcal{C}_{pr})$ is inherited from $\mathcal{L}$.

Example 3. *The operational resolution of Example 1 is canonical if and only if $\Sigma = \mathcal{L} \setminus \{0\}$.*

For any operational resolution $\mathcal{C}_{pr} : \mathcal{P}(\Sigma) \to \mathcal{L}$, we can define a canonical resolution $\mathcal{C}'_{pr} : \mathcal{P}(\Sigma') \to \mathcal{L}'$ such that there exists a map $\phi : \Sigma \to \Sigma'$ fulfilling $\forall T \in \mathcal{P}(\Sigma): \mathcal{C}_{pr}(T) = \mathcal{C}'_{pr}(\{\phi(t) \mid t \in T\})$, that is:

$$
\begin{array}{ccccc}
& \Sigma & & \mathcal{P}(\Sigma) & \xrightarrow{\mathcal{C}_{pr}} & im(\mathcal{C}_{pr}) \\
exists & \phi \downarrow & such\ that & \mathcal{P}(\phi) \downarrow & & || & commutes, \\
& \Sigma' & & \mathcal{P}(\Sigma') & \xrightarrow{\mathcal{C}'_{pr}} & im(\mathcal{C}'_{pr})
\end{array}
$$

where $\mathcal{P}(\phi)(T) = \{\phi(t) \mid t \in T\}$. Indeed, an obvious example of such a construction is the following:

$$
\begin{cases} \Sigma' = im(\mathcal{C}_{pr}) \setminus \{\mathcal{C}_{pr}(\emptyset)\} \\ \mathcal{L}' = im(\mathcal{C}_{pr}) \end{cases} \qquad
\begin{cases} \phi : \Sigma \to \Sigma' : t \to \mathcal{C}_{pr}(\{t\}) \\ \mathcal{C}'_{pr} : \mathcal{P}(\Sigma') \to \mathcal{L}' : T \mapsto \vee T \end{cases}
$$

Moreover, such a canonical resolution is determined up to an isomorphism of its domain and a choice for $\mathcal{L}'$. Indeed, let $\mathcal{C}''_{pr} : \mathcal{P}(\Sigma'') \to \mathcal{L}''$ be another canonical resolution determined by $\mathcal{C}_{pr} : \mathcal{P}(\Sigma) \to \mathcal{L}$, then by definition $\Sigma'' \cup \{0''\} \cong im(\mathcal{C}''_{pr}) = im(\mathcal{C}_{pr}) = im(\mathcal{C}'_{pr}) \cong \Sigma' \cup \{0'\}$, thus there is a bijection $\xi : \Sigma' \to \Sigma''$ which implies that there is an isomorphism of lattices $\mathcal{P}(\xi) : \mathcal{P}(\Sigma') \xrightarrow{\sim} \mathcal{P}(\Sigma'')$. We formulate the net result of the above reasoning as a theorem.

Theorem 2. *Every operational resolution defines an essentially unique canonical one.*

3. STATE TRANSITIONS AS MORPHISMS

Consider the collection of all $(\Sigma, \mathcal{L}, \mathcal{C}_{pr})$ such that $\mathcal{C}_{pr} : \mathcal{P}(\Sigma) \to \mathcal{L}$ is an operational resolution. This will be the object collection of a first category. To deal with the morphisms between two such triples $(\Sigma_1, \mathcal{L}_1, \mathcal{C}_{pr,1})$ and $(\Sigma_2, \mathcal{L}_2, \mathcal{C}_{pr,2})$ we first introduce the following notations, applying on maps $f : \mathcal{P}(\Sigma_1) \to \mathcal{P}(\Sigma_2)$:

$A_\cup$: $\forall \{T_i\}_i \subseteq \mathcal{P}(\Sigma_1) : f(\cup_i T_i) = \cup_i f(T_i)$;
$A_\emptyset$: $\forall T \in \mathcal{P}(\Sigma_1) : f(T) = \emptyset \Leftrightarrow T = \emptyset$;
$A_\#$: $\forall T, T' \in \mathcal{P}(\Sigma_1) : \mathcal{C}_{pr,1}(T) = \mathcal{C}_{pr,1}(T') \Rightarrow \mathcal{C}_{pr,2}(f(T)) = \mathcal{C}_{pr,2}(f(T'))$.

Mathematically, these three conditions encode the "structure preserving" nature of a map $f : \mathcal{P}(\Sigma_1) \to \mathcal{P}(\Sigma_2)$ with respect to the objects 'operational resolutions' expressed as triples $(\Sigma, \mathcal{L}, \mathcal{C}_{pr})$, in a way that will become clear in Proposition 2.

Proposition 1. *We can define a quantaloid $\underline{Res}_\emptyset^\#$, in which the join of maps is computed pointwise, by taking as object class the collection of triples $(\Sigma, \mathcal{L}, \mathcal{C}_{pr})$ such that $\mathcal{C}_{pr} : \mathcal{P}(\Sigma) \to \mathcal{L}$ is an operational resolution, and taking as hom-set between any two such objects $(\Sigma_1, \mathcal{L}_1, \mathcal{C}_{pr,1})$ and $(\Sigma_2, \mathcal{L}_2, \mathcal{C}_{pr,2})$:*

$$\{f : \mathcal{P}(\Sigma_1) \to \mathcal{P}(\Sigma_2) \mid im(f) = \emptyset \ or \ f \ meets \ A_\cup, A_\emptyset, A_\#\}$$

Proof: (o) In this proof, as in all others, the cases where the "bottom" morphism, given by the underlying map $\emptyset : \mathcal{P}(\Sigma_1) \to \mathcal{P}(\Sigma_2) : T \mapsto \emptyset$, comes into play are trivial, so we do not consider them. But it is important to include this map in each hom-set for these to have a bottom element. (i) Identity morphisms — morphisms of which the underlying map is the identity — obviously meet all three conditions, and for composable morphisms f_1, f_2 — morphisms with composable underlying maps — the composite meets all conditions: $(f_2 \circ f_1)(\cup_i T_i) = f_2(\cup_i(f_1(T_1))) = \cup_i f_2(f_1(T_i) = \cup_i(f_2 \circ f_1)(T_i)$, also $(f_2 \circ f_1)(T) = \emptyset \Leftrightarrow f_1(T) = \emptyset \Leftrightarrow T = \emptyset$ and finally $\mathcal{C}_{pr,1}(T) = \mathcal{C}_{pr,1}(T') \Rightarrow \mathcal{C}_{pr,2}(f_1(T)) = \mathcal{C}_{pr,2}(f_1(T')) \Rightarrow \mathcal{C}_{pr,3}((f_2 \circ f_1)(T)) = \mathcal{C}_{pr,3}((f_2 \circ f_1)(T'))$. (ii) Pointwise joins of maps exist: for morphisms $f_i : (\Sigma_1, \mathcal{L}_1, \mathcal{C}_{pr,1}) \to (\Sigma_2, \mathcal{L}_2, \mathcal{C}_{pr,2})$ we have: $(\bigvee_i f_i)(\cup_j T_j) = \cup_{i,j} f_i(T_j) = \cup_j (\bigvee_i f_i)(T_j)$ and $(\bigvee_i f_i)(T) = \emptyset \Leftrightarrow \cup_i f_i(T) = \emptyset \Leftrightarrow \forall i : f_i(T) = \emptyset \Leftrightarrow T = \emptyset$ and finally also $\mathcal{C}_{pr,1}(T) = \mathcal{C}_{pr,1}(T') \Rightarrow \forall i : \mathcal{C}_{pr,2}(f_i(T)) = \mathcal{C}_{pr,2}(f_i(T'))$ which implies $\mathcal{C}_{pr,2}((\bigvee_i f_i)(T)) = \mathcal{C}_{pr,2}(\cup_i f_i(T)) = \vee_i \mathcal{C}_{pr,2}(f_i(T)) = \vee_i \mathcal{C}_{pr,2}(f_i(T')) = \mathcal{C}_{pr,2}(\cup_i f_i(T')) = \mathcal{C}_{pr,2}((\bigvee_i f_i)(T'))$. (iii) Consider a collection of 'parallel' morphisms $f_i : (\Sigma_1, \mathcal{L}_1, \mathcal{C}_{pr,1}) \to (\Sigma_2, \mathcal{L}_2, \mathcal{C}_{pr,2})$ and a composable morphism $g : (\Sigma_2, \mathcal{L}_2, \mathcal{C}_{pr,2}) \to (\Sigma_3, \mathcal{L}_3, \mathcal{C}_{pr,3})$, then we have that:

$(g \circ (\bigvee_i f_i))(-) = g(\cup_i(f_i(-))) = \cup_i(g(f_i(-))) = \bigvee_i(g \circ f_i)(-)$. Likewise for the distributivity on the right. $\diamond\diamond\diamond$

First note that we had to "add" a bottom element to our collection of maps in order to obtain a complete lattice — this bottom element is the empty union of maps. When we interpret the above maps as state transitions, something that will be discussed in detail bellow, this bottom element in a lattice of state transitions "plays the same role" as the bottom element in a property lattice: the latter stands for the "absurd property" which a system will never have; as such can the zero map be interpreted as an "absurd transition". In the next section we will reconsider this aspect and show that it makes sense to introduce "partially absurd transitions".

The very idea behind "operational resolution" — assigning to any subset T of a system's state set Σ a strongest property of that system implied by all the states in T — suggests that any morphism $f \in \underline{Res}_{\emptyset}^{\#}((\Sigma_1, \mathcal{L}_1, \mathcal{C}_{pr,1}), (\Sigma_2, \mathcal{L}_2, \mathcal{C}_{pr,2}))$ is "reflected" through the given operational resolutions as:

$$f_{pr} : im(\mathcal{C}_{pr,1}) \to im(\mathcal{C}_{pr,2}) : \mathcal{C}_{pr,1}(T) \mapsto \mathcal{C}_{pr,2}(f(T)),$$

yielding exactly the corresponding 'definite property transition'. Indeed, if the strongest — definite — actual property of a system, initially in a state that is contained in a $T \subseteq \Sigma$, is $\mathcal{C}_{pr,1}(T) \in im(\mathcal{C}_{pr,1})$, then after the "change of state" f the state of the system is in $f(T)$, thus with strongest — definite — actual property $\mathcal{C}_{pr,2}(f(T))$. It is then exactly condition $A_{\#}$ on the morphism f that assures us that f_{pr} is well-defined: for $a \in im(\mathcal{C}_{pr,1})$ the value of $f_{pr}(a)$ does not depend on the "representative" $T \in \Sigma_1$ that we choose such that $\mathcal{C}_{pr,1}(T) = a$. In other terms, it implies that the following diagram commutes:

$$
\begin{array}{ccc}
\mathcal{P}(\Sigma_1) & \xrightarrow{\ f\ } & \mathcal{P}(\Sigma_2) \\
\mathcal{C}_{pr,1} \downarrow & & \downarrow \mathcal{C}_{pr,2} \\
im(\mathcal{C}_{pr,1}) & \xrightarrow{\ f_{pr}\ } & im(\mathcal{C}_{pr,2})
\end{array}
$$

Further it can be verified that such an f_{pr}, which is a map between join complete lattices, preserves joins: $f_{pr}(\bigvee_i \mathcal{C}_{pr,1}(T_i)) = f_{pr}(\mathcal{C}_{pr,1}(\cup_i T_i)) = \mathcal{C}_{pr,2}(f(\cup_i T_i)) = \mathcal{C}_{pr,2}(\cup_i f(T_i)) = \bigvee_i \mathcal{C}_{pr,2}(f(T_i)) = \bigvee_i f_{pr}(\mathcal{C}_{pr,1}(T_i))$. Taking into account the arguments in the introduction on the propagation of properties we can interpret these formal results.

Physically conclusive: Conditions $A_{\#}$ and $A_{\cup}$ assure that a 'possible state transition' $f : \mathcal{P}(\Sigma_1) \to \mathcal{P}(\Sigma_2)$ determines a unique join preserving 'definite property transition' $f_{pr} : im(\mathcal{C}_{pr,1}) \to im(\mathcal{C}_{pr,2})$.

Note that f_{pr} maps the bottom of its domain exactly onto the bottom of its codomain: $f_{pr}(\mathcal{C}_{pr,1}(T)) = 0_2 \Leftrightarrow \mathcal{C}_{pr,2}(f(T)) = 0_2 \Leftrightarrow f(T) = \emptyset \Leftrightarrow$

$T = \emptyset \Leftrightarrow \mathcal{C}_{pr,1}(T) = 0_1$. We give these conditions applying on a map $g : im(\mathcal{C}_{pr,1}) \to im(\mathcal{C}_{pr,2})$ a notation:

$A_\vee$: $\forall \{a_i\}_i \subseteq im(\mathcal{C}_{pr,1}) : g(\vee_i a_i) = \vee_i g(a_i)$;
A_0: $\forall a \in im(\mathcal{C}_{pr,1}) : g(a) = 0_1 \Leftrightarrow a = 0_2$.

Proposition 2. *We can define a quantaloid $\underline{Res}_0$, in which the join of maps is computed pointwise, by taking as object class the collection of triples $(\Sigma, \mathcal{L}, \mathcal{C}_{pr})$ such that $\mathcal{C}_{pr} : \mathcal{P}(\Sigma) \to \mathcal{L}$ is an operational resolution, and taking as hom-set between any two such objects $(\Sigma_1, \mathcal{L}_1, \mathcal{C}_{pr,1})$ and $(\Sigma_2, \mathcal{L}_2, \mathcal{C}_{pr,2})$:*

$$\{f : im(\mathcal{C}_{pr,1}) \to im(\mathcal{C}_{pr,2}) \mid im(f) = \{0\} \text{ or } f \text{ meets } A_\vee, A_0\}$$

And the following action on an object $(\Sigma, \mathcal{L}, \mathcal{C}_{pr})$ and a morphism $f : (\Sigma_1, \mathcal{L}_1, \mathcal{C}_{pr,1}) \to (\Sigma_2, \mathcal{L}_2, \mathcal{C}_{pr,2})$ in $\underline{Res}_0$ defines a full bijective quantaloid morphism $F_{pr} : \underline{Res}_\emptyset^\# \to \underline{Res}_0$:

$$\begin{cases} F_{pr}(\Sigma, \mathcal{L}, \mathcal{C}_{pr}) = (\Sigma, \mathcal{L}, \mathcal{C}_{pr}) \\ F_{pr}(f) = f_{pr} : im(\mathcal{C}_{pr,1}) \to im(\mathcal{C}_{pr,2}) : \mathcal{C}_{pr,1}(T) \mapsto \mathcal{C}_{pr,2}(f(T)) \end{cases}$$

Proof: We leave the straightforward verification that $\underline{Res}_0$ is a quantaloid to the reader. The action of F_{pr} on objects is simply the identity, so nothing to verify there. The above remarks point out that the action on morphisms is well-defined, and that indeed any f_{pr} is a morphism of $\underline{Res}_0$. We now prove functorality: Since the underlying map of an identity morphism is an identity, it follows that F_{pr} preserves identities, and pasting together commutative diagrams:

$$\begin{array}{ccccc} \mathcal{P}(\Sigma_1) & \xrightarrow{f_1} & \mathcal{P}(\Sigma_2) & \xrightarrow{f_2} & \mathcal{P}(\Sigma_3) \\ \mathcal{C}_{pr,1} \downarrow & & \downarrow \mathcal{C}_{pr,2} & & \downarrow \mathcal{C}_{pr,3} \\ im(\mathcal{C}_{pr,1}) & \xrightarrow{f_{1,pr}} & im(\mathcal{C}_{pr,2}) & \xrightarrow{f_{2,pr}} & im(\mathcal{C}_{pr,3}) \end{array}$$

yields $F_{pr}(f_2 \circ f_1) = F_{pr}(f_2) \circ F_{pr}(f_1)$. F_{pr} induces $\vee$-preserving maps on hom-sets: consider $f_i : (\Sigma_1, \mathcal{L}_1, \mathcal{C}_{pr,1}) \to (\Sigma_2, \mathcal{L}_2, \mathcal{C}_{pr,2})$ in $\underline{Res}_\emptyset^\#$, then $(\vee_i f_i)_{pr}(\mathcal{C}_{pr,1}(T)) = \mathcal{C}_{pr,2}(\cup_i f_i(T)) = \vee_i \mathcal{C}_{pr,2}(f_i(T)) = \vee_i(f_{i,pr}(\mathcal{C}_{pr,1}(T)))$ $= (\vee_i f_{i,pr})(\mathcal{C}_{pr,1}(T))$. Further consider the following situation, where g is a given $\underline{Res}_0$-morphism:

$$\begin{array}{ccc} \mathcal{P}(\Sigma_1) & \xRightarrow{\exists? g^*} & \mathcal{P}(\Sigma_2) \\ \mathcal{C}_{pr,1} \downarrow & & \downarrow \mathcal{C}_{pr,2} \\ im(\mathcal{C}_{pr,1}) & \xrightarrow{g} & im(\mathcal{C}_{pr,2}) \end{array}$$

Remembering the factorization of an operational resolution, *in casu* $\mathcal{C}_{pr,2} = \theta_2 \circ \mathcal{C}_2$, it makes sense to define:

$$g^* : \mathcal{P}(\Sigma_1) \to \mathcal{P}(\Sigma_2) : T \mapsto \cup_{t \in T}(\theta_2^{-1} \circ g \circ \mathcal{C}_{pr,1})(t)$$

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and thus g^* is (the underlying map of) a morphism from $(\Sigma_1, \mathcal{L}_1, \mathcal{C}_{pr,1})$ to $(\Sigma_2, \mathcal{L}_2, \mathcal{C}_{pr,2})$ in $\underline{Res}_\emptyset^\#$: $g^*(\cup_i T_i) = \cup_i g(T_i)$ is obvious; $g^*(T) = \emptyset \Leftrightarrow$ $\forall t \in T : (\theta_2^{-1} \circ g \circ \mathcal{C}_{pr,1})(t) = \emptyset \Leftrightarrow \forall t \in T : g(\mathcal{C}_{pr,1}(t)) = 0_2 \Leftrightarrow \forall t \in T :$ $\mathcal{C}_{pr,2}(t) = 0_1 \Leftrightarrow T = \emptyset$; the following square commutes:

$$
\begin{array}{ccc}
\mathcal{P}(\Sigma_1) & \xrightarrow{\ g^*\ } & \mathcal{P}(\Sigma_2) \\
\mathcal{C}_{pr,1} \downarrow & & \downarrow \mathcal{C}_{pr,2} \\
im(\mathcal{C}_{pr,2}) & \xrightarrow{\ g\ } & im(\mathcal{C}_{pr,1})
\end{array}
$$

since $\mathcal{C}_{pr,2}(g^*(t)) = (\mathcal{C}_{pr,2} \circ \theta_2^{-1} \circ g \circ \mathcal{C}_{pr,1})(t) = g(\mathcal{C}_{pr,1}(t))$ for any $t \in \Sigma_1$, which implies that for any $T \subseteq \Sigma_1$ also $\mathcal{C}_{pr,2}(g^*(T)) = g(\mathcal{C}_{pr,1}(T))$. This proves at once $A_\#$ and $F_{pr}(g^*) = g$. Thus F_{pr} restricted to hom-sets is surjective. $\diamond\diamond\diamond$

Physically conclusive: 'possible state transitions' and implied 'definite property transitions' are in categorical correspondence under the binary operation 'composition of maps' that formally implements consecution of transitions.

We will now comment on the construction in the proof of the "reciprocal" g^* for a given g in $\underline{Res}_0$. Since the restriction of F_{pr} to hom-sets:

$$
F_{pr} \quad : \underline{Res}_\emptyset^\#(-,-) \to \underline{Res}_0(F_{pr}(-), F_{pr}(-))
$$

is join-preserving, it has a unique meet-preserving Galois dual [4,17,21]:

$$
F_{pr}^* \quad : \underline{Res}_0(F_{pr}(-), F_{pr}(-)) \to \underline{Res}_\emptyset^\#(-,-) :
$$
$$
g \mapsto \bigvee \{ f \in \underline{Res}_\emptyset^\#(-,-) \mid F_{pr}(f) \le g \}
$$

We can show the following.

Remark 1. *Referring to the above notations we have:*
(i) $g^* = F_{pr}^*(g)$ for any $g \in \underline{Res}_0(F_{pr}(-), F_{pr}(-))$;
(ii) $F_{pr} \circ F_{pr}^* = id : \underline{Res}_\emptyset^\#(-,-) \to \underline{Res}_\emptyset^\#(-,-) : f \mapsto f$;
(iii) F_{pr}^* preserves composition;
(iv) in general F_{pr}^* is not functoral;
(v) in general F_{pr}^* does not preserve arbitrary joins.

Proof: (i) Since $g^* \in \{f \in \underline{Res}_\emptyset^\#(-,-) \mid F_{pr}(f) \le g\}$, $g^* \le F_{pr}^*(g)$ is obvious. For any $f \in \underline{Res}_\emptyset^\#(-,-)$ such that $F_{pr}(f) \le g$ and any $t \in \Sigma_1$ we have that $f(t) \subseteq (\theta_2^{-1} \circ \mathcal{C}_{pr,2} \circ f)(t) \subseteq (\theta_2^{-1} \circ g \circ \mathcal{C}_{pr,1})(t) = g^*(t)$, therefore, for any $T \subseteq \Sigma_1$, also $f(T) = \cup_{t \in T} f(t) \subseteq \cup_{t \in T} g^*(t) = g^*(T)$. In other terms, $f \le g^*$ for any such f, thus $F_{pr}^*(g) \le g^*$. (ii) Corollary of (i). (iii) Setting that $F_{pr}^*(g_2 \circ g_1) = F_{pr}^*(g_2) \circ F_{pr}^*(g_1)$ we see that

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$F_{pr}(F^*_{pr}(g_2) \circ F^*_{pr}(g_1)) = F_{pr}(F^*_{pr}(g_2)) \circ F_{pr}(F^*_{pr}(g_1)) = g_2 \circ g_1$. (iv) for $id : im(\mathcal{C}_{pr}) \to im(\mathcal{C}_{pr})$, $F^*_{pr}(id) : \mathcal{P}(\Sigma) \to \mathcal{P}(\Sigma) : T \mapsto \cup\{\mathcal{C}(t) \mid t \in T\}$, which in general contains T but is not necessarily contained in T — so identities are not preserved. (v) All we know is that F^*_{pr} preserves meets, nothing more. $\diamond\diamond\diamond$

Conclusion: this Galois dual — in fact, this right inverse — to F_{pr} cannot be extended to a functor, let alone a quantaloid morphism. However, from Proposition 2 we can derive the following equivalence, to be understood as the categorization of Theorem 2.

Proposition 3. *Set that* $U : \underline{Res}_0 \to \underline{JCLat}_0$ *works on* $(\Sigma, \mathcal{L}, \mathcal{C}_{pr})$ *and* $f \in \underline{Res}_0((\Sigma_1, \mathcal{L}_1, \mathcal{C}_{pr,1}), (\Sigma_2, \mathcal{L}_2, \mathcal{C}_{pr,2}))$ *respectively as:*

$$\begin{cases} U(\Sigma, \mathcal{L}, \mathcal{C}_{pr}) = im(\mathcal{C}_{pr}) \\ U(f) = f : im(\mathcal{C}_{pr,1}) \to im(\mathcal{C}_{pr,2}) : \mathcal{C}_{pr,1}(T) \mapsto \mathcal{C}_{pr,2}(f(T)) \end{cases}$$

This defines a fully faithful surjective quantaloid morphism.

Proof: U is surjective on objects because for any given $\mathcal{L}$, object of $\underline{JCLat}_0$, $\mathcal{C}_{pr} : \mathcal{P}(\mathcal{L} \setminus \{0\}) \to \mathcal{L} : T \mapsto \vee T$, cfr. Example 3, has as image through U exactly $\mathcal{L}$. The rest is trivial. $\diamond\diamond\diamond$

Corollary 1. *The quantaloids $\underline{Res}_0$ and $\underline{JCLat}_0$ are equivalent. Moreover, $(U^* \circ U)(\Sigma, \mathcal{C}_{pr}, \mathcal{L})$ — where $U^* : \underline{JCLat} \to \underline{Res}_0$ is the functor that together with U constitutes the equivalence of $\underline{JCLat}$ and $\underline{Res}_0$ — is the essentially unique canonical resolution determined by $(\Sigma, \mathcal{C}_{pr}, \mathcal{L})$.*

Proof: Surjectivity on objects implies that V is isomorphism-dense[3]. A functor that is full, faithful and isomorphism-dense describes the equivalence of its domain and codomain [1,5,19]. $\diamond\diamond\diamond$

By Theorem 1, an equivalent characterization for $\mathcal{C}_{pr} : \mathcal{P}(\Sigma) \to \mathcal{L}$ is given by $(\Sigma, \mathcal{L}, \mathcal{C}, \theta)$ such that $\mathcal{C}_{pr} = \theta \circ \mathcal{C}$. As such, the collection of all operational resolutions gives rise to a bijective collection of such quadruples. The following lemma shows that the morphisms between operational resolutions can be characterized with the aid of only the closure-part of the respective operational resolutions.

Lemma 1. *For a map $f : \mathcal{P}(\Sigma_1) \to \mathcal{P}(\Sigma_2)$ meeting $A_\cup$ and two operational resolutions $\mathcal{C}_{pr,i} = \theta_i \circ \mathcal{C}_i : \mathcal{P}(\Sigma_i) \to \mathcal{L}_i$ there is an equivalence of condition $A_\#$ with:*

A_*: $\forall T \in \mathcal{P}(\Sigma_1) : f(\mathcal{C}_1(T)) \subseteq \mathcal{C}_2(f(T))$.

Proof: $A_\#$ implies A_*: $\mathcal{C}_{pr,1}(T) = \mathcal{C}_{pr,1}(\mathcal{C}_1(T)) \Rightarrow \mathcal{C}_{pr,2}(f(T)) = \mathcal{C}_{pr,2}(f(\mathcal{C}_1(T))) \Rightarrow \mathcal{C}_2(f(T)) = \mathcal{C}_2(f(\mathcal{C}_1(T))) \Rightarrow f(\mathcal{C}_1(T)) \subseteq \mathcal{C}_2(f(T))$. Conversely, A_* implies $A_\#$: $\mathcal{C}_{pr,1}(T') = \mathcal{C}_{pr,1}(T) \Rightarrow \mathcal{C}_1(T') = \mathcal{C}_1(T) \Rightarrow f(\mathcal{C}_1(T')) = f(\mathcal{C}_1(T)) \Rightarrow \mathcal{C}_2(f(\mathcal{C}_1(T'))) = \mathcal{C}_2(f(\mathcal{C}_1(T))) \Rightarrow \mathcal{C}_2(f(T')) = \mathcal{C}_2(f(T)) \Rightarrow \mathcal{C}_{pr,2}(f(S)) = \mathcal{C}_{pr,2}(f(T))$. $\diamond\diamond\diamond$

Proposition 4. *We can define a quantaloid $\underline{Res}_\emptyset^*$, in which joins of maps are computed pointwise, by taking as object class the collection of quadruples $(\Sigma, \mathcal{L}, \mathcal{C}, \theta)$ such that $\mathcal{C}_{pr} = \theta \circ \mathcal{C} : \mathcal{P}(\Sigma) \to \mathcal{L}$ is an operational resolution, and taking as hom-set between any two such objects $(\Sigma_1, \mathcal{L}_1, \mathcal{C}_1, \theta_1)$ and $(\Sigma_2, \mathcal{L}_2, \mathcal{C}_2, \theta_2)$:*

$$\{f : \mathcal{P}(\Sigma_1) \to \mathcal{P}(\Sigma_2) \mid im(f) = \emptyset \text{ or } f \text{ meets } A_\cup, A_\emptyset, A_*\}.$$

Further, $\underline{Res}_\emptyset^\#$ and $\underline{Res}_\emptyset^*$ are isomorphic categories.

The following is obvious.

Proposition 5. *Setting that $V : \underline{Res}_\emptyset^* \to \underline{Clos}_\emptyset$ works on objects $(\Sigma, \mathcal{L}, \mathcal{C}, \theta)$ and $f \in \underline{Res}_\emptyset^*((\Sigma_1, \mathcal{L}_1, \mathcal{C}_1, \theta_1), (\Sigma_2, \mathcal{L}_2, \mathcal{C}_2, \theta_2))$ as:*

$$\begin{cases} V(\Sigma, \mathcal{L}, \mathcal{C}, \theta) = (\Sigma, \mathcal{C}) \\ V(f) : \mathcal{P}(\Sigma_1) \to \mathcal{P}(\Sigma_2) : T \mapsto f(T) \end{cases}$$

defines a fully faithful surjective quantaloid morphism.

Proof: $V(\Sigma, \mathcal{L}, \mathcal{C}, \theta)$ is a closure space for which $\mathcal{C}(\emptyset) = \emptyset$. Surjectivity on objects: for a $\underline{Clos}_\emptyset$-object $(X, \mathcal{C})$ we have evidently that $V(X, \mathcal{F}(X), \mathcal{C}, id_{\mathcal{F}(X)}) = (X, \mathcal{C})$, cfr. Example 2. V is the "identity" on underlying maps of morphisms so nothing to verify there. $\diamond\diamond\diamond$

Corollary 2. $\underline{Res}_\emptyset^*$ *and* $\underline{Clos}_\emptyset$ *are categorically equivalent.*

Using the material of the appendix, we can summarize this categorical setting by the following scheme of quantaloids and quantaloid morphisms:

$$\begin{array}{ccc} \underline{Res}_\emptyset^\# & \xleftrightarrow{\ iso\ } & \underline{Res}_\emptyset^* \\ \downarrow & & \updownarrow \cong \\ \underline{Res}_0 & & \underline{Clos}_\emptyset \\ \cong \updownarrow & & \downarrow \\ \underline{JCLat}_0 & \hookleftarrow & \underline{Intsys}_\emptyset \end{array}$$

4. EXTENDING TO RESOLUTION OPERATORS

In the foregoing, it is apparent that the third condition in the definition of 'operational resolution' — see Eq.(4) — has for consequences that:

- the closure factor of an operational resolution, cfr. Theorem 1, is such that $\mathcal{C}(\emptyset) = \emptyset$;

- the morphisms in the category $\underline{Res}_\emptyset^\#$ must satisfy $A_\emptyset$ in order to be structure preserving; likewise for the morphisms in the category $\underline{Res}_\emptyset^*$; likewise for the morphisms in the category $\underline{Clos}_\emptyset$ to make Proposition 5 work;

- the morphisms in the category $\underline{Res}_0$ must satisfy A_0 in order to make Proposition 2 work; likewise for the morphisms in the category $\underline{JCLat}_0$ to make Proposition 3 work.

Its motivation is primarily that of a conservation law, expressing that whenever we have a physical system in some state beforehand, we still have a physical system in some state after a possible transition. However, one easily verifies that in our constructions, we only need this "empty kernel condition" to prove other "empty kernel conditions". In other words, we can develop a completely analogous scheme without this condition, giving rise to more general objects and more general morphisms. Referring to the previous section, the as such included transitions with non-empty kernels can be interpreted as "partially absurd transitions" where only the subset of the state space assuring a non-empty image is of physical relevance. Purely mathematical, this construction extends in this way the morphismsets of the categories in [14,20] — see also our appendix on this aspect.

Definition 4. *A 'resolution operator' from a set Σ to a poclass $(\mathcal{L}, \leq)$ is a map $\mathcal{R} : \mathcal{P}(\Sigma) \to \mathcal{L}$ such that for all $T, T', T_i \in \mathcal{P}(\Sigma)$:*

$$T \subseteq T' \qquad \Rightarrow \qquad \mathcal{R}(T) \leq \mathcal{R}(T') \qquad (5)$$
$$\forall i \in I : \mathcal{R}(T_i) \leq \mathcal{R}(T) \quad \Rightarrow \quad \mathcal{R}(\cup_i T_i) \leq \mathcal{R}(T) \qquad (6)$$

The image of $\mathcal{R}$ is a complete join semilattice with $\vee_i \mathcal{R}(T_i) = \mathcal{R}(\cup_i T_i)$, a generating set $\{\mathcal{R}(t) \mid t \in \Sigma\}$, as bottom $\mathcal{R}(\emptyset)$ and as top $\mathcal{R}(\Sigma)$. We have the following result in analogy to Theorem 1.

Theorem 3. *Any resolution operator $\mathcal{R} : \mathcal{P}(\Sigma) \to \mathcal{L}$ factors uniquely into a closure operator $\mathcal{C} : \mathcal{P}(\Sigma) \to \mathcal{P}(\Sigma)$ on Σ, and a po-inclusion of the $\mathcal{C}$-closed subsets $\mathcal{F}(\Sigma)$ into $\mathcal{L}$, say $\theta : \mathcal{F}(\Sigma) \hookrightarrow \mathcal{L}$, such that $\mathcal{F}(\Sigma) \cong im(\theta)$.*

Note that a resolution operator $\mathcal{R}$ is a T_0-resolution (T_1) if and only if the closure factor $\mathcal{C}$ is so. For any two given resolution operators $\mathcal{R}_1 = \theta_1 \circ \mathcal{C}_1 : \mathcal{P}(\Sigma_1) \to \mathcal{L}_1$ and $\mathcal{R}_2 = \theta_2 \circ \mathcal{C}_2 : \mathcal{P}(\Sigma_2) \to \mathcal{L}_2$, we recall for a map $f : \mathcal{P}(\Sigma_1) \to \mathcal{P}(\Sigma_2)$:

$A_\cup$: $f(\cup_i T_i) = \cup_i f(T_i)$;
$A_\#$: $\mathcal{R}_1(T) = \mathcal{R}_1(T') \Rightarrow \mathcal{R}_2(f(T)) = \mathcal{R}_2(f(T'))$
A_*: $\forall T \subseteq \Sigma_1 : f(\mathcal{C}_1(T)) \subseteq \mathcal{C}_2(f(T))$

and for $g : im(\mathcal{R}_1) \to im(\mathcal{R}_2)$:

$A_\vee$: $f(\vee_i a_i) = \vee_i f(a_i)$.

We have the following results in analogy to Propositions 1, 2 and 4 and Corollaries 1 and 2.

Proposition 6. *(i) We can define a quantaloid $\underline{Res}^{\#}$ with as objects resolution operators, written as triples $(\Sigma, \mathcal{L}, \mathcal{R})$, and with morphisms $f : (\Sigma_1, \mathcal{L}_1, \mathcal{R}_1) \to (\Sigma_2, \mathcal{L}_2, \mathcal{R}_2)$ determined by corresponding underlying maps $f : \mathcal{P}(\Sigma_1) \to \mathcal{P}(\Sigma_2)$ that meet $A_\cup$ and $A_\#$. The join of morphisms is computed pointwise.*
(ii) We can define a quantaloid $\underline{Res}$ with as objects resolution operators, again written as triples $(\Sigma, \mathcal{L}, \mathcal{R})$, and with morphisms $g : (\Sigma_1, \mathcal{L}_1, \mathcal{R}_1) \to (\Sigma_2, \mathcal{L}_2, \mathcal{R}_2)$ determined by corresponding underlying maps $g : im(\mathcal{R}_1) \to im(\mathcal{R}_2)$ that meet $A_\vee$. The join of morphisms is computed pointwise. Setting for an object $(\Sigma, \mathcal{L}, \mathcal{R})$ and a morphism $f : (\Sigma_1, \mathcal{L}_1, \mathcal{R}_1) \to (\Sigma_2, \mathcal{L}_2, \mathcal{R}_2)$ of $\underline{Res}^{\#}$ that:

$$\begin{cases} F_\mathcal{R}(\Sigma, \mathcal{L}, \mathcal{R}) = (\Sigma, \mathcal{L}, \mathcal{R}) \\ F_\mathcal{R}(f) : im(\mathcal{R}_1) \to im(\mathcal{R}_2) : \mathcal{R}_1(T) \mapsto \mathcal{R}_2(f(T)) \end{cases}$$

defines a full bijective quantaloid morphism $F_\mathcal{R} : \underline{Res}^{\#} \to \underline{Res}$. Further, $\underline{Res}$ is equivalent to $\underline{JCLat}$.
(iii) We can define a quantaloid $\underline{Res}^*$ with as objects resolution operators, now written as quadruples $(\Sigma, \mathcal{L}, \mathcal{C}, \theta)$, and with morphisms $f : (\Sigma_1, \mathcal{L}_1, \mathcal{C}_1, \theta_1) \to (\Sigma_2, \mathcal{L}_2, \mathcal{C}_2, \theta_2)$ determined by corresponding underlying maps $f : \mathcal{P}(\Sigma_1) \to \mathcal{P}(\Sigma_2)$ that meet $A_\cup$ and A_*. The join of morphisms is computed pointwise. This quantaloid is isomorphic to $\underline{Res}^{\#}$ and equivalent to $\underline{Clos}$.

To summarize:

$$
\begin{array}{ccc}
\underline{Res}^{\#} & \xleftrightarrow{\;iso\;} & \underline{Res}^* \\
\downarrow & & \updownarrow{\cong} \\
\underline{Res} & & \underline{Clos} \\
{\cong}\updownarrow & & \downarrow \\
\underline{JCLat} & \hookleftarrow & \underline{Intsys}
\end{array}
$$

which is exactly the same scheme that closed the previous subsection, however without the "empty kernel conditions".

5. CONCLUSION: ON POSSIBLE STATE TRANSITIONS

In the introduction we already sketched the reasoning in [15] which assures that properties propagate with preservation of the join. In that same paper it is shown that with almost no requirements it is possible to derive the unitary evolution of a particle if one assumes strong determinism, i.e., when f sends states — being the atoms of the supposedly complete atomistic orthomodular property lattice — onto states. However, this hypothesis of "strong deterministic evolution" disables us to

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express indeterministic transitions that do occur when considering for example a perfect quantum measurement of the property a and its orthocomplement a', where the propagation of the 'possible states' is described by the following map:

$$f : \mathcal{P}(\Sigma) \to \mathcal{P}(\Sigma) : \left\{ \begin{array}{l} \{p\} \mapsto \{a \wedge (a' \vee p), a' \wedge (a \vee p)\} \\ T \mapsto \cup\{f(\{p\}) \mid p \in T\} \end{array} \right.$$

provided that $T \cup \{0\}$ is interpreted as 'possible states' T since 0 is never true. This map sends any possible initial state on its two possible outcome states $a \wedge (a' \vee p)$ and $a' \wedge (a \vee p)$, formally expressed as f being the union of the maps that are atomically generated[4] by the respective Sasaki projectors. Clearly f cannot be "reduced" to a join preserving map between property lattices, but one can verify that it does satisfy our definition of a state transition — one can indeed prove that any union of maps that are atomically generated by join preserving maps satisfies $A_{\#}$ and $A_{\cup}$. Since this particular state transition occurs in standard quantum theory, being an ordinary measurement described by a self-adjoint operator with eigenspaces corresponding to a and a', it clearly cannot suffice to work in a mathematical category where the morphisms representing state transitions are join preserving maps between the atomistic property lattices, as is implicitly the case in [15,29]. In the case of our — quantaloid — duality of categories, one category has the 'physically justifiable definite property transitions' — described by join preserving maps between property lattices — as morphisms, and the other category has the 'underlying possible state transitions' as morphisms, all this allowing the description of indeterministic evolutions and as such generalizing the strong deterministic evolutions to arbitrary ones. Within this context we also mention an application related to linear logic [16,32], which provides a syntactical tool to describe the above mentioned perfect quantum measurement of a and a' [10]. More general, the mathematical scheme presented in this paper delivers a class of semantical interpretations for the corresponding logic that describes the process of indeterministic propagation of states for entities with a not necessarily Boolean description. As a present topic of further study we also mention the implications of aspects of weak modularity and orthocomplementation within our scheme, considering what already has been done for general algebraic quantales [22,23,24].

6. DISCUSSION: DESCRIBING COMPOUNDNESS

In this section we discuss the use of the morphismsets in our dual quantaloids for the description of compound systems in the spirit of [6,7,8]. In particular, with the hypothesis that any kind of interaction between two systems boils down to the fact that 'actuality of a property a_1 of the

first system is caused by to actuality of property a_2 of the second system', we can recover the rays of the tensor product of Hilbert spaces as the description of the 'states of compoundness' for corresponding quantum systems [9]. It can indeed again be argued that also in this case, the corresponding maps — describing mutual induction of properties — should be join preserving: with any two interacting physical systems, respectively described by lattices of verifiable properties $\mathcal{L}_1$ and $\mathcal{L}_2$, we can associate a map $f^* : \mathcal{L}_2 \to \mathcal{L}_1$ with $a_1 = f^*(a_2)$ being the cause of a_2 in $\mathcal{L}_1$, that is, a_1 is the weakest property in $\mathcal{L}_1$ whose actuality causes the actuality of a_2; all this again assures a join preserving Galois dual f that expresses mutual induction of properties of one system onto the other. Then again, by applying the same tools as in [15], i.e., the general theory of morphisms of projective geometries [14] in combination with Piron's representation theorem [25,26], it is possible to prove that one obtains a complete lattice, with the anti-Hilbert-Schmidt maps as atoms — the atoms of the obtained complete lattice are exactly the join preserving maps that send atoms on atoms or the bottom element — and $[\mathcal{L}_1 \setminus \{0_1\} \to \mathcal{L}_2 : a_1 \mapsto 1_2; 0_1 \mapsto 0_2]$ as top element [9]:

(i) We can interpret the anti-Hilbert-Schmidt maps between Hilbert space H_1 and H_2 as 'states of maximal compoundness' since they correspond in a one to one way with the rays in $H_1 \otimes H_2$.

(ii) The top element can be interpreted as the 'state of separation' — differently discussed in [2], where separation refers to a type of entity and not to a state of a compound system — since it expresses that actuality of a property of the first system implies "existence" of the second and — strictly — nothing more, this assuming that both systems exist *a priori*.

As such, the morphismsets in our quantaloids generalize the description of the interaction between individual entities within compound systems. One can proceed the same reasoning for general compound systems consisting of any number of individual entities. It follows that a general description for compound systems corresponds with a commuting diagram within the two dual quantaloids: the arrows in the diagram represent the mutual induction of states and properties and the commutativity follows from the requirement that there should be 'structural independence' on the order of performance of the measurements on the individual entities within the compound system. A paper on the matter is in the publication pipeline.

APPENDIX: QUANTALOIDS

Below we give some mathematical preliminaries to the content of this paper related to quantaloids. References are [1,5,19] for categories and [18,28,31] for quantaloids.

Definition 5. *A quantaloid is a category such that:*
(i) every hom-set is a join complete semilattice;
(ii) composition of morphisms distributes on both sides over joins.
Let Q and R be quantaloids. A quantaloid morphism from Q to R is a functor $F : Q \to R$ such that on hom-sets it induces join-preserving maps $Q(A, B) \to R(FA, FB)$.

In the language of enriched category theory [5] we can say that a quantaloid is a category that is enriched in *JCLat*, the category of join complete semilattices and join-preserving maps, and a quantaloid morphism is a *JCLat*-enriched functor. A quantaloid with one object is commonly known as a 'unital quantale' [22,30]. Another point of view is that in a quantaloid every hom-set of endomorphisms on an object (a hom-set of "loops") is a unital quantale. The restriction of a quantaloid morphism to a hom-set of 'loops' yields what is known as a 'unital quantale morphism'. The quantaloids that are constructed in this paper, are exactly in this way generalizations of the unital quantales that are constructed, and motivated physically, in [3,11].

Example 4. *The category of join complete semilattices and join-preserving maps JCLat is a quantaloid, with respect to pointwise ordering of maps [28].*

As can easily be verified, any subcategory of a quantaloid that is closed under the inherited join of morphisms, is a subquantaloid. Thus any full subcategory of a quantaloid is a subquantaloid, and selecting from a given a quantaloid certain morphisms but keeping all the objects, gives rise to a subquantaloid if and only if the inherited join of morphisms is internal. Often, such a subquantaloid is constructed by imposing extra conditions on the morphisms, verifying that these extra conditions "respect" arbitrary joins.

Example 5. *Selecting from JCLat those morphisms $f : \mathcal{L} \to \mathcal{M}$ that meet the extra condition $f(a) = 0_{\mathcal{M}} \Leftrightarrow a = 0_{\mathcal{L}}$, we obtain a new quantaloid since any join of such maps is again such a map. We will denote this new quantaloid by $JCLat_0$.*

Example 6. *Consider a category with as objects closure spaces $(X, \mathcal{C})$, in which a morphism $f : (X_1, \mathcal{C}_1) \to (X_2, \mathcal{C}_2)$ is represented by an underlying union-preserving map between the respective powersets, that is, $f : \mathcal{P}(X_1) \to \mathcal{P}(X_2)$ such that $f(\cup_i T_i) = \cup_i f(T_i)$. This is in fact a quantaloid in which the join of maps is computed pointwise. Keeping all the objects and selecting those morphisms that satisfy $\forall T \in \mathcal{P}(X_1) : f(\mathcal{C}_1(T)) \subseteq \mathcal{C}_2(f(T))$, we obtain a subquataloid that we will denote by Clos, since the condition respects the join of morphisms. Now selecting those closure spaces $(X, \mathcal{C})$ for which $\mathcal{C}(\emptyset) = \emptyset$ and those morphisms that send $\emptyset$ exactly on $\emptyset$, that is, $f(T) = \emptyset \Leftrightarrow T = \emptyset$, we obtain a subquantaloid $Clos_\emptyset$ of Clos, since the extra condition on morphisms respects joins.*

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Another category of closure spaces that plays an important role in for instance [14,15,20] has as objects all closure spaces $(X, \mathcal{C})$ and as morphisms between $(X_1, \mathcal{C}_1)$ and $(X_2, \mathcal{C}_2)$ all of the 'continuous' maps $f : X_1 \setminus K \to X_2$ defined on the complement of $K \subseteq X_1$, that is $f(\mathcal{C}_1(T) \setminus K) \subseteq \mathcal{C}_2(f(T \setminus K))$ for all $T \subseteq X$. Denoting this category as $\underline{Space}$, it is easy to see that there is a functor $Ext : \underline{Space} \to \underline{Clos}$ that is the identity on objects and:

$$Ext(f) : \mathcal{P}(X_1) \to \mathcal{P}(X_2) : T \mapsto \{f(x) | x \in T \setminus K\}$$

for a morphism $f : (X_1, \mathcal{C}_1) \to (X_2, \mathcal{C}_2)$. But the morphisms $Ext(f)$ meet the extra condition that $Ext(f)(\{x\})$ is a singleton or the empty set for all $x \in X$. Since this condition is not preserved by joins, $\underline{Space}$ is not a quantaloid and Ext is not a quantaloid morphism. However, $\underline{Space}$ can be embedded in $\underline{Clos}$ and this embedding restricts to an embedding $\underline{Space}_\emptyset \to \underline{Clos}_\emptyset$, where $\underline{Space}_\emptyset$ is the category with those objects of $\underline{Space}$ such that $\mathcal{C}(\emptyset) = \emptyset$ and of which all morphisms have an empty kernel. The same sort of remark can be made on the categories of lattices. The typical category of lattices that one finds in [14,15,20] is $\underline{JCALat}$: its objects are complete atomistic lattices, its morphisms are join complete lattices that send atoms onto atoms or onto the bottom (the full subcategory $\underline{T_1Space}$ of $\underline{Space}$ consisting of T_1-closures is then equivalent to $\underline{JCALat}$ — which is exactly the core of the mathematical developments in both [14,20]). $\underline{JCALat}$ can be embedded into $\underline{JCLat}$ as category, simply by a "forgetful functor" $U : \underline{JCALat} \to \underline{JCLat}$, but again it is evidently not true that the join of morphisms $U(f_i)$ is a morphism $U(f)$, because such a join does not necessarily send atoms onto atoms. Of course, this embedding restricts to an embedding $U : \underline{JCALat}_0 \to \underline{JCLat}_0$, where now $\underline{JCALat}_0$ is the subcategory of $\underline{JCALat}$ of which the morphisms never send an atom to the bottom.

Example 7. *By an 'intersection system' we mean a collection of subsets of a certain set X, ordered by inclusion, closed under intersection. Any intersection system is a complete lattice, ordered by set-inclusion, thus we can define a quantaloid $\underline{Intsys}$ as the full subcategory of $\underline{JCLat}$ of which the objects are intersection systems. Accordingly we construct $\underline{Intsys}_\emptyset$ as the full subcategory of $\underline{JCLat}_0$ of which the objects are intersection systems with bottom element $\emptyset$.*

Example 8. *Setting that $W : \underline{Clos} \to \underline{Intsys}$ works on $(\Sigma, \mathcal{C})$ and $f \in \underline{Clos}((\Sigma_1, \mathcal{C}_1), (\Sigma_2, \mathcal{C}_2))$ respectively as:*

$$\begin{cases} W(\Sigma, \mathcal{C}) = \mathcal{F}(\Sigma) \\ W(f) : \mathcal{F}_1(\Sigma_1) \to \mathcal{F}_2(\Sigma_2) : F \mapsto \mathcal{C}_2(f(F)) \end{cases}$$

we have defined a full bijective quantaloid morphism. The same is true for the obvious sub-functor $W : \underline{Clos}_\emptyset \to \underline{Intsys}_\emptyset$

Proof: Since a closure operator $\mathcal{C}$ on a set Σ can be characterized completely by the intersection system $\mathcal{F}(\Sigma)$ of $\mathcal{C}$-closed subsets, the bijectivity is clear. To show that the action of W is functoral and full requires some verifications that are analogous to those of the proof of Proposition 2. $\diamond\diamond\diamond$

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NOTES

1. This formulation — discussed with D.J. Moore privately — differs from the one in [15] the sense that it does not make any reference to the tests that define properties in an operational way [2,15,20,25,26].

2. 'Poclass' is short for 'partially ordered class', being a thin category $\mathcal{L}$ in which any two different objects are non-isomorphic, wherein we write $a \leq b$ if and only if there is (exactly) one morphism from a to b. Since Definition 1 makes no reference to the whole of $\mathcal{L}$ but only to at most set-many elements of $\mathcal{L}$, we can indeed work with a poclass rather than a poset for the codomain $\mathcal{L}$. The partial ordering on the codomain $\mathcal{L}$ can be operationally motivated [26]. The fact that $\mathcal{L}$ might be larger than $im(\mathcal{C}_{pr})$ is essential: we need to be able to consider one $\mathcal{L}$ for different Σ's and $\mathcal{C}_{pr}$'s, with not coinciding images, allowing the joint consideration of the properties of a compound system and those of its subsystem.

3. A functor $F : \underline{A} \to \underline{B}$ is called isomorphism-dense if for any $\underline{B}$-object B there exists some $\underline{A}$-object A such that $F(A) \cong B$.

4. A map is atomically generated when the image of $T \subseteq \Sigma$ is the union of the images of $p \in \Sigma$ by the underlying atomic map, in this case a Sasaki projection. For details we refer to [3].